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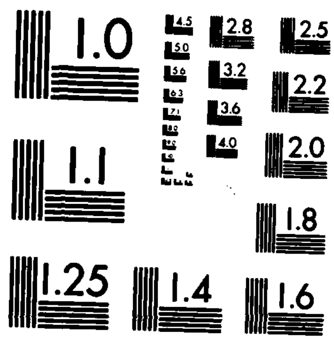

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APPROXIMATING THE PROBABILITIES OF DETECTING AND OF HITTING TARGETS AND THE PROBABILITY DISTRIBUTION OF THE NUMBER OF TRIALS ALONG THE VISIBLE PORTIONS OF CURVES IN THE PLANE SUBJECT TO A POISSON SHADOWING PROCESS*

by

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State University of New York at Binghamton

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Key Words: Poisson Shadowing Process, Bernoulli Trials, Visibility Probabilities, τ-reduced measure of Visibility, Detection Probability, Hitting Probability.

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0. Introduction

A hunter is trying to detect and hit a deer in a forest. Suppose that a deer is moving along a path in the forest and the hunter is located among the trees at some distance from the path. The path is only partially visible to the hunter; the invisible (shadowed) portion of the path is obscured by the trees which are dispersed randomly between the hunter and the path. A deer can be detected by the hunter if at least a certain part of it is visible. This occurs only if at least one of the visible segments of the path is sufficiently long.

After detection of a deer, in a visible segment, the hunter starts shooting. The deer continues, however, to move along the path in the same pace. During each shooting trial the deer crosses a length of τ of the path. Thus the number of shooting trials in each visible segment depends on the length of the segment. The shooting trials stop when the deer is either hit or enters an invisible portion of the path. When the deer enters another visible segment, it has to be detected agin. For simplicity we assume that the shooting trials are Bernoulli, with probability of failure q, $0<q<1$.

The problem of deer hunting can be treated as a two dimensional shadowing problem. The hunter is located at a point $\underset{\sim}{0}$ in the plane, the deer moves along a curve C in the plane, and the trunks of trees can be described as random disks dispersed between $\underset{\sim}{0}$ and C. Two dimensional random shadowing problems were previously studied

by Chernoff and Daly [1]. Likhterov and Gurin [2], Yadin and Zakcs [3,4]. The methodology developed in the present paper is also applicable to three dimensional versions of the above problem. For example, if a hunter tries to shoot down a bird whose flying course is partially obscured by crowns of trees. The three dimensional shadowing problem was previously studied by Yadin and Zacks [5].

In the present study we develop approximations for (a) the probability of detection; (b) the probability distribution of the maximal number of shooting trials N; and (c) the probability of survival of the deer (bird). We also provide numerical examples to illustrate the goodness of these approximations.

In section 1 we review the notions and definitions concerning random Poisson shadowing fields in the plane, measures of random visibility on star shaped curves, and introduce the notion of τ-reduced measures of visibility. This notion enables us to define detection probabilities in terms of the distribution of the τ-reduced measure of visibility. In addition we can also arrive at lower bounds to the number of Bernoulli trials, N, along the visible portion of a curve. The moment generating function (MGF) of the τ-reduced measure of visibility yields lower and upper bounds to the probabilities of a successful hunt. Section 2 discusses the moments of the τ-reduced visibility measure and introduce an approximation to its distribution. With the aid of this approximation we obtain bounds for the MGF mentioned above.

Section 3 and 4 present approximations to the cumulative distribution functions of the τ-reduced visibility measure $V(\tau)$, and of the number of trials, N, along the visible portions. Section 5 is devoted to the problem of determining the probability of simultaneous visibility of n points along the curve C, and the recursive determination of the moments of $V(\tau)$. Section 6 provides some special cases of two-dimensional models. Finally, in Section 7 we provide a numerical example. The methodology presented here for two-dimensional shadowing problems can be generalized in a straight forward manner to three-dimensional, by applying similar modifications to the results of [5].

1. The Model, Measures of Visibility and Failure Probabilities

Suppose that the hunter is located at the origin, $\underset{\sim}{0}$, and let C denote the path of the deer. C is assumed to be a smooth star shaped curve, defined by a piece-wise differentiable function $r(s)$, $s_L \le s \le s_u$, representing the distance from $\underset{\sim}{0}$ to C in orientation s. The polar coordinates of a point $\underset{\sim}{P}_s$ on C are $(r(s),s)$. The end-points of C are $\underset{\sim}{P}_{s_L}$ and $\underset{\sim}{P}_{s_u}$. The length of C is

$$(1.1) \qquad L = \int_{s_L}^{s_u} \ell(s)ds \qquad ,$$

where

$$\ell(s) = [r^2(s) + (\frac{d}{ds} r(s))^2]^{1/2}$$

The trees in the forest are presented by random disks dispersed in a region between $\underset{\sim}{0}$ and C. Each random disk is characterized

by coordinates (ρ,θ,y), where (ρ,θ) are the polar coordinates of its center and y is its diameter. The coordinates (ρ,θ,y) belong to a set S in R^3 satisfying conditions which assure that $\underset{\sim}{0}$ is not covered and C is not intersected by random disks. Let $\mathcal{B}$ be the Borel σ-field on the sample space S, and let $N\{B\}$ designate the number of disks whose coordinates belong to a set B of $\mathcal{B}$. We assume that, for each $B\varepsilon\mathcal{B}$, $N\{B\}$ is a random variable having a Poisson distribution with mean

$$(1.2) \qquad \nu\{B\} = \lambda \iiint_B H(d\rho,d\theta)dG(y|\rho,\theta),$$

where $G(y|\rho,\theta)$ is the conditional CDF of y, given (ρ,θ), and $H(d\rho,d\theta)$ is a σ-finite measure of (ρ,θ). Such a random field of disks is called a <u>Poisson random field</u>.

A point $\underset{\sim}{P}_s$ on C is said to be <u>visible</u> if the line segment $\overline{\underset{\sim}{O}\underset{\sim}{P}_s}$ is not intersected by any random disk. A point which is not visible is in a shadow. The measure of total visibility on C is defined as

$$(1.3) \qquad V = \int_{s_L}^{s_u} I(s)\ell(s)ds \qquad ,$$

where $I(s)=1$ if $\underset{\sim}{P}_s$ is visible, and $I(s)=0$ otherwise. Notice that V is a random variable representing the total length of the visible portion of C. V is a sum of a random number, M, of visible segments of C having random length $X_1, X_2, \ldots, X_M$; i.e.

$$(1.4) \qquad V = \sum_{i=1}^{M} X_i \quad .$$

A target is detected only if there exists at least one visible segment of length greater than the minimal path length τ_o required for identifying the target. In order to develop a formula for the probability of detecting a target, we introduce the notion of <u>τ-reduced</u> <u>visibility</u> <u>measure</u>, $V(\tau)$, which is the total length of visible segments, each one reduced by τ units, i.e.,

$$(1.5) \qquad V(\tau) = \sum_{i=1}^{M} (X_i - \tau)_+$$

where $a_+ = \max(a,0)$. The probability that a target is not detected is

$$(1.6) \qquad p_o(\tau_o) = \Pr\{V(\tau_o) = 0\}.$$

On the other hand, the probability that C is completely visible is

$$(1.7) \qquad p_1 = \Pr\{V(\tau) = L-\tau\}, \text{ for all } 0 \le \tau \le L.$$

Indeed, when C is completely visible, M=1 and X_1=L. Let N denote the number of shooting trials, after detecting a target. If a single shooting trial requires a segment of length τ to be completely visible, then

$$(1.8) \qquad N = \sum_{i=1}^{M} [(X_i - \tau_o)_+/\tau] \quad ,$$

where [a] is the maximal integer not exceeding a. Notice that

$$(1.9) \qquad \frac{1}{\tau}\sum_{i=1}^{M}(X_i-\tau_o-\tau)_+ \le N \le \frac{1}{\tau}\sum_{i=1}^{M}(X_i-\tau)_+$$

Hence, according to (1.5) and (1.9),

$$(1.10) \qquad V(\tau_1)/\tau \le N \le V(\tau_o)/\tau \quad ,$$

where $\tau_1 = \tau_o + \tau$.

If the probability of failure in each shooting trial is q, and the shooting trials are independent (Bernoulli), the number of shooting trials required until the first success, J, is distributed geometrically. Accordingly, the probability of failure (not hitting the deer) is $Q = E\{q^N\}$. Thus, according to (1.10), lower and upper bounds for Q are, respectively, Q_o and Q_1, where

$$(1.11) \qquad Q_i = E\{q^{V(\tau_i)/\tau}\} \quad , i = 0,1$$

Notice that Q_i is the value of the MGF of $V(\tau_i)$ at the point $t = (\log q)/\tau$.

2. The Moments and Moment Generating Function of $V(\tau)$.

For the sake of determing the moments of $V(\tau)$ we introduce the following definition of this measure,

$$(2.1) \qquad V(\tau) = \int_{s_{L,\tau}}^{s_{u,\tau}} I_\tau(s)\,\ell(s)ds \quad ,$$

where $I_\tau(s) = 1$ if a segment of C of length τ, centered at $(r(s),s)$ is completely visible, and $I_\tau(s) = 0$ otherwise. $s_{L,\tau}$ and $s_{U,\tau}$ are direction coordinates of points within C, of distance $\tau/2$ along C from s_L and s_U respectively. More formally, let

$$(2.2) \qquad L(s) = \int_{s_L}^{s} \ell(y)dy.$$

Then, $s_{L,\tau} = L^{-1}(\tau/2)$ and $s_{U,\tau} = L^{-1}(L-\tau/2)$.

The n-th moments of $V(\tau)$ is thus

$$\eta_n(\tau) = E\{(\int_{s_{L,\tau}}^{s_{U,\tau}} I_\tau(s)\ell(s)ds)^n\}$$

$$(2.3) \qquad = n! \int_{A_{n,\tau}} \cdots \int E\{\prod_{i=1}^{n} I_\tau(s_i)\} \prod_{i=1}^{n} \ell(s_i)ds_i.$$

The set $A_{n,\tau}$ is the simplex

$$(2.4) \qquad A_{n,\tau} = \{(s_1, \ldots, s_n); \ s_{L,\tau} \le s_1 < \ldots < s_n \le s_{U,\tau}\} \quad .$$

Furthermore, $E\{\prod_{i=1}^{n} I_\tau(s_i)\}$ is the probability that the union of n segments of C, each one of length τ, centered at n points having direction coordinates $s_1 < \ldots < s_n$, is completely visible. This probability is designated by $p_n(s_1, \ldots, s_n; \tau)$. Thus the n-th moment of $V(\tau)$ is

$$(2.5) \qquad \eta_n(\tau) = n! \int_{A_{n,\tau}} \cdots \int p_n(s_1, \ldots, s_n; \tau) \prod_{i=1}^{n} \lambda(s_i) ds_i .$$

The method for determining $p_n(s_1, \ldots, s_n; \tau)$ and $\mu_n(\tau)$ is based on a general methodology developed by Yadin and Zacks [3,4] for the special case of $\tau=0$. The main results of those papers, with the modifications required for $\tau>0$, are presented in Sections 4 and 5.

3. An Approximation to the CDF of $V(\tau)$

The cumulative distribution function (CDF) of $V(\tau)$ is a mixture of a two-point distribution concentrated on $\{0, L-\tau\}$ and a distribution concentrated on the interval $(0, L-\tau)$. For the purpose of presenting the approximation discussed below, we consider a normalized measure of visibility $W(\tau) = V(\tau)/(L-\tau)$, which is concentrated on $[0,1]$. The CDF of $W(\tau)$ can be represented as

$$(3.1) \qquad F_\tau(w) = \begin{cases} 0 & , \text{ if } w<0 \\ p_o(\tau)+(1-p_o(\tau)-p_1)\, F_\tau^*(w), & 0 \le w < 1 \\ 1 & , \quad 1 \le w \end{cases} .$$

If, for example, $G(y|\rho,\theta)$ is absolutely continuous then $F_\tau^*(w)$ is an absolutely continuous CDF on $(0,1)$. Let $\mu_n(\tau)$ denote the n-th moment of $W(\tau)$. Obviously, $\eta_n(\tau) = (L-\tau)^n \mu_n(\tau)$, $n=1,2,\ldots$.

Furthermore,

$$(3.2) \qquad \mu_n(\tau) = p_1 + (1-p_o(\tau)-p_1) \int_0^1 w^n dF_\tau^*(w) \quad , \quad n = 1, 2, \ldots \quad .$$

Applying the Dominated Convergence Theorem one immediately proves that $\lim_{n\to\infty} \mu_n(\tau) = p_1$ for all $\tau \geq 0$.

Explicit expressions for $p_o(\tau)$ and $F_\tau^*(w)$ are not available. We apply here a beta approximation to $F_\tau^*(w)$ and provide a numerical approximation to $p_o(\tau)$. This type of mixed-beta approximation was applied also in [3,4,5]. As will be shown in Section 6, in some special cases, the first ten moments of $W(\tau)$ and of the mixed-beta approximation are very close. This indicates that in those cases one has a highly effective approximation. In cases where the moments are not in agreement better approximation should be attempted. The approximating beta-mixture CDF is given by the formula

$$(3.3) \qquad \tilde{F}_\tau(w) = \begin{cases} 0 & , \text{ if } w<0 \\ \tilde{p}_o(\tau) + (1-\tilde{p}_o(\tau)-p_1) I_w(\alpha_\tau, \beta_\tau), & 0 \leq w < 1 \\ 1 & , \quad \text{if } 1 \leq w \ , \end{cases}$$

where $I_w(\alpha,\beta)$, $0 \leq w \leq 1$, $0<\alpha,\beta<\infty$, denotes the incomplete beta function ratio. The probability of complete visibility of C, p_1, is determined by the shadowing model, as shown later. The values of $\tilde{p}_o(\tau)$, α_τ and β_τ are determined by equating the formulae of the first three moments of $\tilde{F}_\tau(w)$ to those of $W(\tau)$. As shown in [3] these approximating parameters are given by

$$\alpha_\tau = \frac{2(\mu_2^*(\tau))^2 - \mu_3^*(\tau)(\mu_1^*(\tau) + \mu_2^*(\tau))}{\mu_1^*(\tau)\mu_3^*(\tau) - (\mu_2^*(\tau))^2}$$

(3.4)

$$\beta_\tau = \frac{(\mu_1^*(\tau) - \mu_2^*(\tau))(\mu_2^*(\tau) - \mu_3^*(\tau))}{\mu_1^*(\tau)\mu_3^*(\tau) - (\mu_2^*(\tau))^2}$$

and

(3.5) $$\tilde{p}_o(\tau) = 1 - p_1 - \mu_1^*(\tau)(\alpha_\tau + \beta_\tau)/\alpha_\tau \quad .$$

where

(3.6) $$\mu_n^*(\tau) = \mu_n(\tau) - p_1 \quad , \qquad n=1,2,3 \quad .$$

Notice that $\mu_n^*(\tau) \geq 0$ for all $n=1,2,\ldots$ and that $\mu_n^*(\tau) \geq \mu_{n+1}^*(\tau)$. As revealed in formula (3.4) the solutions for α_τ and β_τ might be sensitive to the numerical accuracy of the moments. $\tilde{p}_o(\tau_o)$ is the mixed-beta approximation to the probability of no detection, $p_o(\tau_o)$.

Likhterov and Gurin [2] developed a theory of random coverage on the real line, according to which $p_o(\tau_o)$ can be obtained by solving certain integral equations. Such a solution requires generally numerical iterative techniques. The method of approximating $p_o(\tau_o)$ from the moments of $W(\tau)$ requires a numerical approximation too, but is simpler to execute.

4. Bounds for the CDF of N and for Q

Inequality (1.10) yields lower and upper bounds for the CDF of N. Indeed, from (1.10),

(4.1) $$F_{\tau_o}\left(\frac{\tau n}{L-\tau_o}\right) \le \Pr\{N \le n\} \le F_{\tau_1}\left(\frac{\tau n}{L-\tau_i}\right)$$

The CDF's in (4.1) can be approximated by the mixed-beta CDF (3.3). According to (1.11), the lower and upper bounds, for the failure probability Q, are the value of the MGF of $W(\tau_i)$ $i=0,1$, at the point $t=\frac{1}{\tau}(L-\tau_i)\log q$. Let $G_\tau(t)$ indicate the MGF of $W(\tau)$. This function can be expressed in terms of the moments of $W(\tau)$ as

(4.2) $$G_\tau(t) = 1 + p_1(e^t-1) + \sum_{n=1}^{\infty} \frac{\mu_n^*(\tau)}{n!} t^n, \quad -\infty<t<\infty .$$

Since $\mu_n^*(\tau) \downarrow 0$ as n grows the infinite series in (4.2) converges faster than e^t, and therefore a small number of terms will often provide a good approximation. Another method of approximating $G_\tau(t)$ is by employing the MGF of the mixed-beta distribution (3.3) with $\tilde{p}(\tau)$, α_τ and β_τ.

5. Visibility Probabilities and Moments for Two Dimensional Models

In Section 2 we introduced the probabilities $p_n(s_1,\ldots,s_n;\tau)$. In the present section we discuss the methodology for determining these functions in two dimensional models, and indicate how the moments can be determined from (2.5) in a recursive fashion. In addition we provide an explicit formula for the determination of the probability of complete visibility, p_1. We start with the presentation of the various functions required for the determination of $p_n(s_1,\ldots,s_n;\ 0)$ and $\mu_n(0)$, and continue with the modification needed for the general case.

Consider the star-shaped curve C, specified by

$$(5.1) \qquad C=\{(\rho,\theta);\ \rho=r(\theta),\ s_L\leq\theta\leq s_u\} \quad ,$$

where $r(\theta)$ is continuous and piece-wise differentiable on $[s_L,s_u]$.

We assume that shadows on the curve C are cast by disks, with centers which are randomly distributed within a strip S, bounded by the curves

$$(5.2) \qquad U = \{(\rho,\theta);\ \rho=u(\theta),\ \theta_L\leq\theta\leq\theta_u\}$$

and

$$(5.3) \qquad W = \{(\rho,\theta);\ \rho=w(\theta),\ \theta_L\leq\theta\leq\theta_u\}$$

where $\theta_L<s_L<s_u<\theta_u$ and $u(\theta)<w(\theta)$. For simplicity of the exposition we consider a standard Poisson shadowing process, according to which the centers of random disks are uniformly distributed in S and their diameters are distributed independently of their location, i.e.

$$(5.4) \qquad H(d\rho,d\theta)dG(y|\rho,\theta) = \rho d\rho d\theta dG(y).$$

In addition, $u(\theta)$, $w(\theta)$ and $G(y)$ should satisfy conditions which insure that the origin $\underset{\sim}{0}$ is uncovered and C is not intersected by random disks. In particular, these conditions require that

$G(y)$ will be concentrated on an interval $[a,b]$ and that the distance of U from $\underset{\sim}{0}$ and of W from C would be at least $b/2$ (see [4]). For the purpose of obtaining the visibility probabilities, we introduce the auxiliary functions

$$(5.5) \qquad K_-(s,t) = \begin{cases} \displaystyle\int_{s-t}^{s} \int_{u(\theta)}^{w(\theta)} G(y(\rho,d-\theta))\rho d\rho d\theta & , t>0 \\ 0 & , t\leq 0 \end{cases}$$

and

$$(5.6) \qquad K_+(s,t) = \begin{cases} \displaystyle\int_{s}^{s+t} \int_{u(\theta)}^{w(\theta)} G(y(\rho,\theta-s))\rho d\rho d\theta & , t>0 \\ 0 & , t\leq 0 \end{cases}$$

where

$$(5.7) \qquad y(\rho,\theta-s) = y(\rho,s-\theta) = \begin{cases} 2\rho\sin|\theta-s| & , \text{ if } |\theta-s|<\pi/2 \\ 2\rho & , \text{ if } |\theta-s|\geq\pi/2 \end{cases}$$

Notice that $y(\rho,\theta-s)$ is the maximal diameter of a disk centered at (ρ,θ) which does not intersect the ray with direction s. Thus, $\lambda K_-(s,t)$ and $\lambda K_+(s,t)$ are the expected number of disks in the region S, having centers with direction coordinates in $[s-t,s]$ and $[s,s+t]$, respectively, which do not intersect the line segment $\overline{\underset{\sim}{O}\underset{\sim}{P}_s}$.

Let $\nu\{S\}$ denote the expected number of disks in the region S, i.e.,

(5.8) $$\nu\{S\}=\lambda \int_{\theta_L}^{\theta_u} \int_{u(\theta)}^{w(\theta)} \rho d\rho d\theta = \frac{\lambda}{2} \int_{\theta_L}^{\theta_u} (w^2(\theta)-u^2(\theta))d\theta.$$

Accordingly, the expected number of disks which can cast shadows on C is

(5.9) $$\nu_1 = \nu\{S\}-\lambda[K_-(s_L,s_L - \theta_L)+K_+(s_u,\theta_u-s_u)] \quad .$$

Thus the probability of complete visibility is $p_1=e^{-\nu_1}$. The probability $p_n(s_1,\ldots,s_n;0)$ fo simultaneous visibility of n points, with $s_L \leq s_1 < \ldots < s_n \leq s_u$ is

(5.10) $$p_n(s_1,\ldots,s_n;0) = \exp\{-\nu\{S\}\}\exp\{\lambda K_-(s_1,s_1-\theta_L)$$
$$+\lambda K_+(s_n,\theta_u-s_n) + \lambda \sum_{i=1}^{n-1} [K_+(s_i,\frac{s_{i+1}-s_i}{2})+K_-(s_{i+1},\frac{s_{i+1}-s_i}{2})]\}$$

Finally, we present the recursive equations for the calculation of the moments $\mu_n(0)$. Define the sequence of functions

(5.11) $$\psi_o(s) = \exp\{\lambda K_-(s,s-s_L)\}$$

and for each $j \geq 1$

(5.12) $$\psi_j(s) = \frac{j}{L} \int_{s_L}^{s} \ell(y)\psi_{j-1}(y)\exp\{\lambda K_-(y,\frac{s-y}{2}) + \lambda K_+(y,\frac{s-y}{2})\}dy.$$

Then, for each $n \geq 1$, the moments of $W(0)$ are

$$(5.13) \qquad \mu_n(0) = \frac{n}{L} \exp\{-\nu\{S\}\} \int_{s_L}^{s_u} \ell(s)\psi_{n-1}(s)\exp\{\lambda K_+(s,\theta_u-s)\}ds.$$

In order to generalize the above results, one has to adjust the functions $K_-(s,t)$ and $K_+(s,t)$ in the following manner. For a given orientation s, define

$$(5.15) \qquad t_-(\tau,s) = s-L^{-1}(L(s)-\tau/2)$$

and

$$(5.16) \qquad t_+(\tau,s) = L^{-1}(L(s)+\tau/2)-s.$$

The function $p_n(s_1,\ldots,s_n;\tau)$ is obtained by substituting in (5.12) $K_-(s-t_-(\tau,s),t-t_-(\tau,s))$ and $K_+(s+t_+(\tau,s),\ t-t_+(\tau,s))$ for $K_-(s,t)$ and $K_+(s,t)$, respectively. Similarly, the moments $\mu_n(\tau)$ are obtained by replacing $K_-(s,t)$ and $K_+(s,t)$ in formulae (5.12) - (5.13) by $K_-(s-t_-(\tau,s),t-t_-(\tau,s))$ and $K_+(s+t_+(\tau,s),t-t_+(\tau,s))$, respectively; and replacing L by $L-\tau$.

6. Some Special Cases of Two Dimensional Models

6.1 Circular Path and Annular Region

The curve C and the region S are specified by the functions

$$u(\theta) = u, \qquad -\frac{\pi}{2} \le \theta_L \le \theta \le \theta_u \le \pi/2$$

$$w(\theta) = w \qquad -\frac{\pi}{2} \le \theta_L \le \theta \le \theta_n \le \pi/2$$

and

$$r(s) = r, \qquad \theta_L < s_L \le s \le s_u < \theta_u \quad .$$

Furthermore, we require that

$$0 < b/2 < u < w < r-b/2$$

Due to the symmetry on the circle, one can show (see [4]) that the K-functions in the annular case do not depend on s, and satisfy

$$(6.1) \qquad K_+(s,t) = K_-(s,t) = K^*(t).$$

Explicit expressions for $K^*(t)$, in the case of a uniform distribution $G(y)$ on $[a,b]$, was derived in [4]. Let $K^*(t) = K_w(t)-K_u(t)$. The functions $K_w(t)$ and $K_u(t)$ can be obtained from

$$K_v(t) = \begin{cases} 0 & , \text{ if } t < \sin^{-1}(\frac{a}{2v}) \\ K_v^{(1)}(t) & , \text{ if } \sin^{-1}(\frac{a}{2v}) \le t < \sin^{-1}(\frac{b}{2v}) \\ K_v^{(2)}(t) & , \text{ if } \sin^{-1}(\frac{b}{2v}) \le t < \pi/2 \\ K_v^{(3)}(t) & , \text{ if } t \ge \pi/2 \end{cases} \quad .$$

where

$$K_v^{(1)}(t) = \frac{1}{b-a}\{\frac{v^2}{3}[(4v^2-a^2)^{1/2} - 2v\cos(t)] -$$

$$\frac{a}{2}v^2[t-\sin^{-1}(\frac{a}{2v})] + \frac{a^2}{24}[(4v^2-a^2)^{1/2} -$$

$$a\ \mathrm{cotan}\ (t)]\},$$

$$K_v^{(2)}(t) = K_v^{(1)}(\sin^{-1}(\frac{b}{2v})) + \frac{v^2}{2}(t-\sin^{-1}(\frac{b}{2v}))$$

$$- \frac{a^2+ab+b^2}{24}[\frac{1}{b}(4v^2-b^2)^{1/2} - \mathrm{cotan}\ (t)],$$

and

$$K_v^{(3)}(t) = K_v^{(2)}(\frac{\pi}{2}) + (t-\frac{\pi}{2})(\frac{v^2}{2} - \frac{a^2+ab+b^2}{24}) \quad .$$

Due to the simple and symmetric nature of C and S, the adjustment factors $t_-(\tau,s)$ and $t_+(\tau,s)$ satisfy

(6.2) $$t_-(\tau,s) = t_+(\tau,s) = \tau/2r \quad .$$

Accordingly, formulae (5.8)-(5.10) are replaced by

(6.3) $$\nu\{S\} = \frac{\lambda}{2}(w^2-u^2)(\theta_u-\theta_L) \quad ,$$

(6.4) $$\nu_1 = \nu\{S\} - \lambda[K^*(s_L-\theta_L)+K^*(\theta_u-s_u)],$$

and

(6.5) $$p_n(s_1,\ldots,s_n;\tau) = \exp\{-\nu\{S\}\}\exp\{\lambda[K^*(s_1-\theta_L-\frac{\tau}{2r}) + K^*(\theta_u-s_n-\frac{\tau}{2r})] + 2\lambda\sum_{i=1}^{n-1} K^*(\frac{s_{i+1}-s_i}{2} - \frac{\tau}{2r})\}.$$

For, determining the moments $\mu_n(\tau)$ we compute recursively,

$$(6.6) \qquad \psi_o(s) = \exp\{\lambda K^*(s-s_L - \frac{\tau}{2r})\} \quad ,$$

and for $j \geq 1$,

$$(6.7) \qquad \psi_j(s) = \frac{rj}{L-\tau} \int_{s_L}^{s} \psi_{j-1}(y) \exp\{2\lambda K^*(\frac{s-y}{2} - \frac{\tau}{2r}) \ dy.$$

Using these functions we obtain,

$$(6.8) \qquad \mu_n(\tau) = \frac{nr\exp\{-\nu\{S\}\}}{L-\tau} \int_{s_L}^{s_u} \psi_{n-1}(y)\exp\{\lambda K^*(\theta_u - y - \frac{\tau}{2r})\}dy$$

6.2 <u>Linear Path C and Trapezoidal Region S.</u>

We consider here the case where both U, W and C are parallel lines of distance u, w and r from $\underset{\sim}{0}$. Furthermore,

$$(6.9) \qquad 0 < \frac{b}{2} \leq u < w < r - \frac{b}{2}$$

In this case it is simple to express the formulae in terms of Cartesian coordinates. Accordingly, a point $\underset{\sim}{P}_s$ on C has the coordinates (s,r). Thus

$$(6.10) \qquad C = \{(s,r)\ ;\ \ s_L \le s \le s_u\} \quad ,$$

$$(6.11) \qquad U = \ (\theta,u)\ ;\ \ \theta_L \le \theta \le \theta_u \quad ,$$

and

$$(6.12) \qquad W = \{(\theta,u)\ ;\ \ \theta_L \le \theta \le \theta_u\} \quad .$$

It was shown in [5] that $K_-(s,t)=K_+(s,t)=K(s,t)$. A formula for $K(s,t)$ is given in [5]. We adopt the convention that $K(s,t)=0$ for all $t \le 0$. For the τ-reduced case we obtain

$$(6.13) \qquad t_-(\tau,s)=t_+(\tau,s)=\tau/2 \quad .$$

Formulae (5.8)-(5.10) assume in the τ-reduced case the form:

$$(6.14) \qquad \nu\{S\} = \frac{\lambda}{2r}\,(\theta_u-\theta_L)(w^2-u^2) \quad ,$$

$$(6.15) \qquad \nu_1 = \nu\{S\} - \lambda[K(s_L,s_L-\theta_L) + K(s_n,\theta_u-s_u)] \quad ,$$

and

$$(6.16) \qquad p_n(s_1,\ldots,s_n;\tau) = \exp\{-\nu\{S\}\} \quad .$$

$$\exp\{\lambda[K(s_1,s_1-\theta_L-\tfrac{\tau}{2}) + K(s_n,\theta_n-s_n-\tfrac{\tau}{2})]+$$

$$\lambda\sum_{i=1}^{n-1}[K(s_i,\tilde{s}_i-\tfrac{\tau}{2}) + K(s_{i+1},\tilde{s}_i-\tfrac{\tau}{2})]\}$$

where

$$(6.17)\qquad \tilde{s}_i = r\tan\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{s_i}{r}\right) + \tan^{-1}\left(\frac{s_{i+1}}{r}\right)\right)\right) \quad .$$

For further explanation, see [5]. Finally, determine recursively the functions

$$(6.18)\qquad \psi_o(s) = \exp\{\lambda K(s, s-\theta_L - \frac{\tau}{2})\}$$

and for $j \geq 1$,

$$(6.19)\qquad \psi_j(s) = \frac{j}{L-\tau} \int_{s_L}^{s} \psi_{j-1}(y) \exp\{2\lambda K(y, \tilde{s}(y) - \frac{\tau}{2})\}\, dy,$$

where

$$(6.20)\qquad \tilde{s}(y) = r\tan\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{s}{r}\right) + \tan^{-1}\left(\frac{y}{r}\right)\right)\right).$$

The moments of $W(\tau)$ are given then by

$$(6.21)\qquad \mu_n(\tau) = \frac{n}{L-\tau}\exp\{-\nu\{S\}\} \int_{s_L}^{s_u} \psi_{n-1}(y) \exp\{\lambda K(y, \theta_u - y - \frac{\tau}{2})\quad dy.$$

7.2 Numerical Example

In the present section we provide an example which demonstrates numerically the results of the present paper. We consider the case of an arc C and annular strip S, which was discussed in Section 6.1. The parameters of this case are:

$\theta_L = -\pi/2$, $s_L = -\pi/3$, $s_u = \pi/3$, $\theta_u = \pi/2$, $r=1$, $w=.6$, $u=.4$, $\lambda=5$.

In addition, the diameters are uniformly distributed over the interval (.1, .5).

In Table 7.1 we present the first 10 moments of $W(\tau)$, for $\tau=0(.1).4$, which have been determined according to (6.6)-(6.8). The corresponding moments of the mixed-beta distribution (3.3) are also given for comparison.

τ \ n	1	2	3	4	5	6	7	8	9	10
0.0	.738 .738	.600 .600	.517 .517	.462 .463	.425 .425	.398 .398	.378 .378	.363 .363	.351 .351	.342 .342
0.1	.704 .704	.561 .561	.479 .479	.427 .427	.393 .393	.369 .369	.351 .351	.338 .338	.329 .329	.321 .321
0.2	.671 .671	.526 .526	.447 .447	.399 .399	.368 .369	.347 .347	.332 .332	.321 .321	.313 .313	.307 .307
0.3	.641 .641	.497 .497	.421 .421	.377 .377	.349 .349	.330 .331	.318 .318	.308 .309	.302 .302	.297 .297
0.4	.614 .614	.471 .471	.399 .399	.359 .359	.334 .334	.318 .318	.307 .307	.299 .300	.294 .294	.289 .290

TABLE 7.1 Moments of $W(\tau)$ (upper line) and of $\tilde{F}_\tau(w)$ (lower line) for $\tau = 0(.1).4$ and $n = 1, \ldots, 10$.

As shown in Table 7.1, the first ten moments obtained from the mixed-beta CDF, $\tilde{F}_\tau(w)$, differ from those obtained by the recursive formulae only at the 4th decimal place. This reveals an excellent approximation to the CDF of $W(t)$ by $\tilde{F}_\tau(w)$, in the case under consideration. In Table 7.2 we provide the parameters of the mixed-beta distributions associated with Table 7.1.

τ	σ	$\tilde{p}_o(t)$	p_1	α_τ	β_τ
0	.2353	.0064	.27	3.3905	1.8888
.1	.2559	.0119	.27	3.0675	2.0334
.2	.2747	.0194	.27	2.8000	2.1640
.3	.2917	.0298	.27	2.6076	2.3093
.4	.3069	.0431	.27	2.4814	2.4808

TABLE 7.2. The Parameters of the Mixed-Beta Distribution $\tilde{F}_\tau(w)$ for $\tau = 0(.1).4$. (σ denotes the standard deviations.)

The values of $\tilde{p}_o(\tau)$ in Table 7.2, provide the mixed-beta approximations to the probabilities $p_o(\tau_o)$ of not detecting a a target. This is obviously an increasing function of τ_o. Thus, in the present example, if $\tau_o = .1$, $\tilde{p}_o(\tau_o) = .012$ while if $\tau_o = .4$, $\tilde{p}_o(\tau_o) = .043$. $p_1 = .27$ is the probability of complete visibility along the path. Since the moments of the mixed-beta distributions $\tilde{F}_\tau(w)$ fitted so well those of $W(\tau)$, we replace $F_{\tau_i}(\frac{\tau n}{L-\tau_i})$ with $\tilde{F}_{\tau_i}(\frac{\tau n}{L-\tau_i})$, $i = 0, 1$. In Table 7.3 we present

$\tilde{F}_{\tau_i}(\frac{\tau n}{L-\tau_i})$ for $\tau_i = 0(.1).4$, $\tau = .1$.

The values of $Q_i=E\{\exp\{t_i W(\tau_i)\}\}$ where $t_i = \frac{s_n - s_L - \tau_i}{\tau} \log(q)$ with q=.8, are also given in Table 7.3.

$n \backslash \tau_i$	0.0	0.1	0.2	0.3	0.4
0	0.0064	0.0119	0.0194	0.0298	0.0431
1	0.0065	0.0122	0.0203	0.0318	0.0466
2	0.0075	0.0147	0.0255	0.0411	0.0613
3	0.0104	0.0211	0.0375	0.0804	0.0894
4	0.0166	0.0332	0.0577	0.0906	0.1307
5	0.0273	0.0520	0.0870	0.1314	0.1836
6	0.0435	0.0785	0.1253	0.1819	0.2458
7	0.0662	0.1131	0.1723	0.2405	0.3145
8	0.0960	0.1557	0.2269	0.3052	0.3865
9	0.1333	0.2059	0.2879	0.3736	0.4585
10	0.1782	0.2628	0.3532	0.4430	0.5272
11	0.2303	0.3252	0.4209	0.5106	0.5894
12	0.2890	0.3914	0.4884	0.5735	0.6423
13	0.3531	0.4592	0.5529	0.6288	0.6836
14	0.4208	0.5261	0.6115	0.6738	0.7117
15	0.4902	0.5891	0.6613	0.7063	0.7265
16	0.5582	0.6448	0.6992	0.7249	1.0000
17	0.6213	0.6896	0.7226	1.0000	1.0000
18	0.8750	0.7183	1.0000	1.0000	1.0000
19	0.7139	1.0000	1.0000	1.0000	1.0000
20	1.0000	1.0000	1.0000	1.0000	1.0000
Q_i	0.0704	0.0967	0.1273	0.1621	0.2000

TABLE 7.3. The CDF $\tilde{F}_{\tau_i}\left(\frac{\tau n}{L-\tau_i}\right)$, with $\tau = .1$, $\tau_i=0(.1).4$, $L=s_u-s_L$; and the corresponding MGF Q_i.

As we see in Table 7.3, if $\tau=.1$ and $\tau_o = .1$ the lower bound of Q is .0967 and the upper bound for Q is .1273. If however, $\tau_o=0$ then $.0704 \leq Q \leq .0967$.

The bounds for the CDF of N are read from Table 7.3 in a similar manner. For example, if $\tau_o=0$, $\tau_1=.1+\tau_o=.1$ then for $n=6$, $.0435 \leq P\{N \leq 6\} \leq .0785$. If, $\tau_o = .1$ then $\tau_1 = .1 + \tau_o = .2$ and $.0785 \leq P\{N \leq 6\} \leq .1253$. Thus, from the first two columns of Table 7.3 we obtain that, when $\tau_o = 0$, the expected number of trials, $E\{N\}$, is between 13.7 and 15.1.

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